CSP for Commutative, Idempotent Groupoids

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Constraint Satisfaction Problem

Definition

An *instance* of the CSP is a triple $\mathcal{R} = (V, \mathbf{A}, \mathcal{C})$ in which:

- V is a finite set of *variables*,
- $\mathbf{A}$ is a finite, idempotent algebra
- $\mathcal{C} = \{(S_i, R_i) \mid i = 1, \ldots, n\}$ is a set of *constraints*, with $S_i \subseteq V$ and $R_i \leq \mathbf{A}^{S_i}$.

A *solution* to $\mathcal{R}$ is a map $f\colon V \to A$ such that for all i, $f(S_i) \in R_i$. The algebra $\mathbf{A}$ is said to be *tractable* if the decision problem CSP($\mathbf{A}$) is in P. A *variety* $\mathcal{V}$ is tractable if every finite algebra in $\mathcal{V}$ is tractable.

Known Results

Theorem (Bulatov and Dalmau)

The variety of quasigroups is tractable.

Definition

An algebra is *congruence meet-semidistributive* (SD($\wedge$)) if its congruence lattice satisfies

$$(x \wedge y \approx x \wedge z) \Rightarrow (x \wedge (y \vee z) \approx x \wedge y)$$

Theorem (Barto and Kozik)

An SD($\wedge$) *variety is tractable.*

Theorem (Jeavons, Cohen, Gyssens '97)

The variety of semilattices is tractable.

The CSP Dichotomy...

Theorem (Bulatov, Jeavons, Krokhin '05; Maroti & McKenzie '08)

Let $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{A}$ *has no weak near-unanimity term (WNU), then* $\mathbf{A}$ *is NP-complete.*

Algebraic Dichotomy Conjecture

If $\mathbf{A}$ has a WNU term, then it is tractable.

Motivation:

- A binary operation is a WNU if and only if is commutative and idempotent.
- Adding associativity suffices for tractability of an algebra.
- Any weakening of associativity should also suffice.

CI-Groupoids

Definition

Let $\mathbf{A} = \langle A, \cdot \rangle$ be a groupoid. We call $\mathbf{A}$ a *CI-groupoid* if $\cdot$ is both commutative and idempotent. Usually, we write xy for $x \cdot y$.

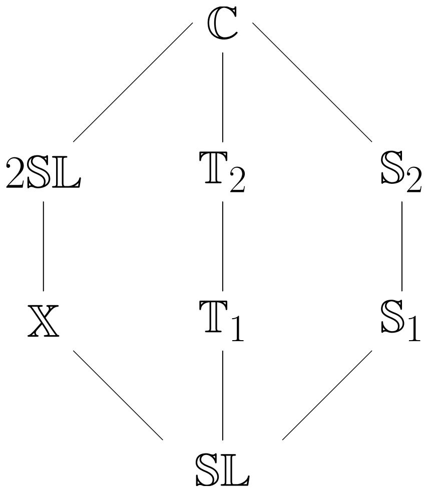
The Moufang Law $x(y(zy)) = ((xy)z)y$ is one weakening of associativity.

Definition

An identity $p \approx q$ is of *Bol-Moufang type* if (i) the only operation in p, q is $\cdot$, (ii) the same three variables appear on both sides, in the same order, (iii) one of the variables appears twice (iv) the remaining two variables appear only once.

- There are 60 such identities. Which ones are equivalent with respect to C+I?

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The Variety $\mathcal{S}_2$ of Bol-Moufang CI-Groupoids

Definition

$\mathcal{S}_2$ is the variety of CI-groupoids satisfying $x(y(xz)) \approx x((yx)z)$.

Theorem (KKVW '13)

A finite idempotent algebra with WNU terms $v(x, y, z)$ and $w(x, y, z, u)$ such that $v(y, x, x) \approx w(y, x, x, x)$ is SD($\wedge$).

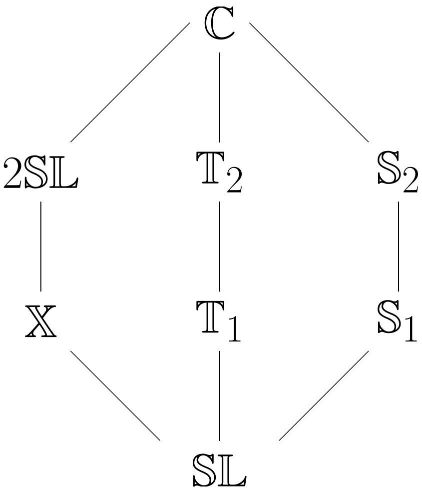
Theorem

$\mathcal{S}_2$ is tractable.

Proof.

$\mathcal{S}_2$ has WNU terms $v(x, y, z) = (xy)(z(xy))$ and $w(x, y, z, u) = (xy)(zu)$ such that $v(y, x, x) \approx w(y, x, x, x)$. □

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The Płonka Sum of Groupoids

Definition

Given

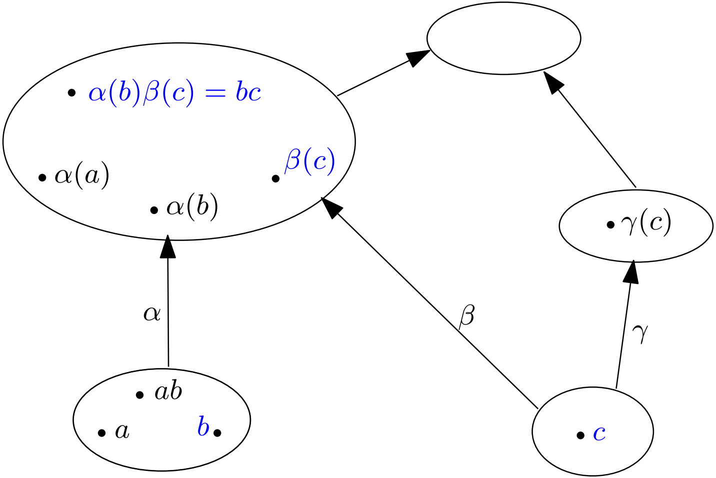
- $\mathbf{S} = \langle S, \vee \rangle$ a semilattice,
- $\{\mathbf{A}_s \mid s \in S\}$ a set of groupoids, and
- $\{\phi_{s,t} : \mathbf{A}_s \to \mathbf{A}_t \mid s \leq_\vee t\}$ a set of "nice" homomorphisms,

the **Płonka sum** over S of the groupoids $\{\mathbf{A}_s : s \in S\}$ is the groupoid $\mathbf{A}$ with universe $\dot{\bigcup}_{s \in S} A_s$ and multiplication given by:

$$x_1 *^{\mathbf{A}} x_2 = \phi_{s_1,s}(x_1) *^{\mathbf{A}_s} \phi_{s_2,s}(x_2)$$

where $x_i \in \mathbf{A}_{s_i}$, $s = s_1 \vee s_2$.

The Płonka Sum of Groupoids



Płonka's Theorem

Theorem

Let $\mathcal{V}$ be the variety of groupoids defined by $\Sigma \cup \{x \vee y \approx x\}$ for some term $x \vee y$ and set Σ of regular identities. The following classes of algebras coincide:

(1) *The class* $\mathbf{Pl}(\mathcal{V})$*of Płonka sums of* $\mathcal{V}$*-algebras.*

(2) *The variety of algebras of type ρ defined by the identities Σ and the following identities:*

$$x \vee x \approx x \tag{P1}$$

$$(x \vee y) \vee z \approx x \vee (y \vee z) \tag{P2}$$

$$x \vee (y \vee z) \approx x \vee (z \vee y) \tag{P3}$$

$$x \vee (y * z) \approx x \vee y \vee z \tag{P4}$$

$$(x * y) \vee z \approx (x \vee z) * (y \vee z) \tag{P5}$$

Pseudopartition Operations

Definition

A term $x \vee y$ satisfying $(P1)$-$(P4)$ is a *pseudopartition operation*. The congruence on an algebra possessing such a term defined by

$$a\,\sigma\,b \Leftrightarrow [a \vee b = a \text{ and } b \vee a = b]$$

is known as the *semilattice replica congruence*.

Theorem (Main Result)

Let $\mathbf{A}$ *be a finite idempotent algebra with pseudopartition operation* $x \vee y$*, such that every block of its semilattice replica congruence lies in the same tractable variety. Then* $\mathbf{A}$ *is tractable.*

Squags and $\mathcal{T}_2$

Definition

$\mathcal{T}_2$ is the variety of CI-groupoids satisfying $x(y(yz)) \approx ((xy)y)z$.

Definition

The variety of Steiner quasigroups (squags) is the variety of CI-groupoids satisfying $y(xy) \approx x$.

Theorem

$\mathcal{T}_2$ is tractable.

Proof.

Let $x \vee y \approx y(xy)$ in $\mathcal{T}_2$. Each σ-class is a squag. □

Theorem

The subvariety $\mathcal{T}_1$ (defined by $x(x(yz)) \approx (x(xy))z$) of $\mathcal{T}_2$ is the class of Płonka sum of squags.

CID and CIE Groupoids

Definition

A groupoid is *distributive (D)* if it satisfies $x(yz) \approx (xy)(xz)$. It is *entropic (E)* if it satisfies $(xy)(zw) \approx (xz)(yw)$.

Theorem

Every finite CID-groupoid (and hence CIE-groupoid) is a Płonka sum of quasigroups.

Corollary

The variety of CID-groupoids is tractable.

Thanks!